Discovery 101

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Abstract: For many undergraduates (including future teachers), geometry is a body of knowledge to be memorized. What happens to this expectation when students are empowered to make authentic mathematical discoveries? This presentation will discuss the use of *The Geometers Sketchpad* as a tool of discovery and its impact on the mathematical thinking, beliefs and expectations of future teachers.

Introduction

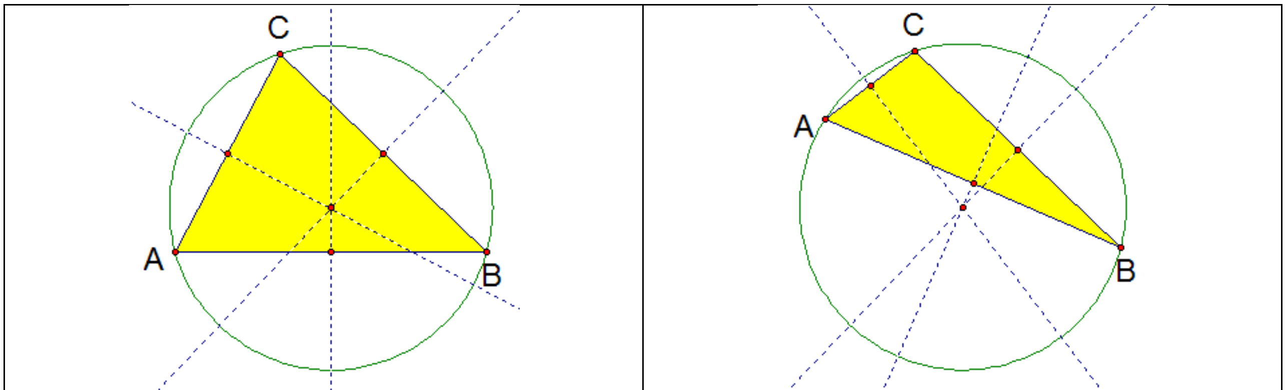
For much of the 20th century, formal instruction in geometry was justified on the grounds that it helped discipline the mind and promote logical reasoning. Consequently, much instructional time and effort was focused on reading and creating proofs. During the 1980s and 1990s, this rationale was supplanted by the Standards movement with its emphasis on connecting mathematical concepts and procedures with those of other disciplines (González & Herbst, 2006). In response to this shift in priorities, instruction at many levels increased emphasis on exploratory activities and problem-solving and decreased emphasis on reasoning and proof.

The Reasoning and Proof Standard is intended to apply throughout the mathematics curriculum, including the study of geometry. From our perspective as mathematics educators, it appears that this standard is seldom implemented as intended. Too many geometry students (including preservice teachers) fail to appreciate the crucial distinction between empirical and deductive arguments and, in general, show a preference for the use of empirical argument over deductive reasoning (Balacheff, 1988; Chazan, 1993; Finlow-Bates, 1994; Martin & Harel, 1989; Porteous, 1990; Williams, 1979).

Hoyles and Jones (1998) recommend using reasoning and proof to communicate, explore and explain geometric concepts and relationships as well as to justify and verify geometric assertions. Devising instructional tasks that do so will require creativity and insight on the part of mathematics educators. Like many, we believe that the most promising context in which to promote this sort of thinking is in the act of discovery. Dynamic geometry provides a supportive environment in which to facilitate discovery and to encourage geometric reasoning.

Discovering Dynamic Geometry

The term *dynamic geometry* refers to geometry done in computer-based modeling environments such as *The Geometers Sketchpad, Cabri Geometry*, *Cinderella* and *Geogebra.* These tools provide intuitive interfaces for constructing, measuring, and transforming geometric objects. Furthermore, as free (i.e., unconstrained) points and lines are repositioned using click-and-drag mouse actions, all displayed measurements and geometric relationships are continuously updated. This capability facilitates one of dynamic geometry's most dramatic uses, the search for and identification of invariant features and/or relationships. For example, Observation 1 in Figure 1 demonstrates a relationship taught in many geometry courses: The perpendicular bisectors of the three sides of a given triangle intersect at a common point called the *circumcenter*. Observation 2 shows that dragging point A to a new location creates a different triangle with the same property. The presence of the same property in both triangles suggests the possibility that this might be a property of other, and possibly all, triangles. This thought may then be expressed as a formal conjecture and tested using the same dynamic geometry construction that gave rise to the conjecture. Testing is accomplished by mouse-dragging free points and lines and observing whether the features and/or relationships of interest are preserved under all testable circumstances.

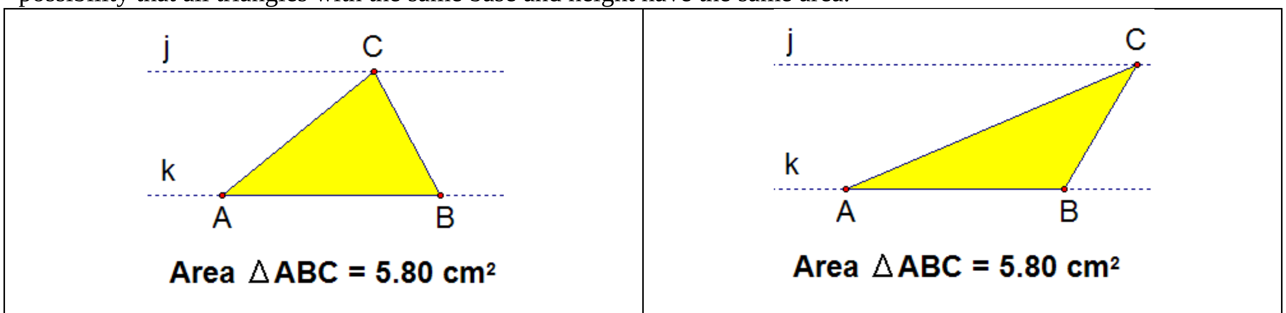


Observation 1: The perpendicular bisectors of the three sides of a given triangle intersect in a common point.

Observation 2: Dragging point A to a new location creates a different triangle with the same property.

Figure 1: Discovering Invariant Features/Relationships

A related use of this approach is seen in Figure 2. In this case, the invariant aspect (i.e., area) of the two figures is not directly observable. But area measurements are observable and comparable. These data suggest the possibility that all triangles with the same base and height have the same area.



Observation 1: The area of !ABC is $5.8cm^2$

Observation 2: The area of !ABC is $5.8cm^2$

Figure 2: Discovering Invariant Measurements/Computations

Using dynamic geometry, discovery may take many forms. For instance, models may be constructed that facilitate the search for minimum or maximum features or values. In Figure 3, the positions of points B and C on the horizontal segment AA" determine the side lengths of !ABC. By adjusting these lengths, one may explore the questions, "For a given perimeter, which triangle has the maximum area?" A similar approach may be used in Figure 4 to answer the question, "For a given perimeter, which rectangle has the maximum area?" In Figure 5, corner squares are cut from a rectangular sheet of paper. The "sides" of the sheet are then folded up to create an open top box. The volume and shape of the box are automatically displayed. By adjusting the size of the corner square, students may explore the question, "What is the maximum volume open top box that may be constructed from a rectangular sheet of paper?"

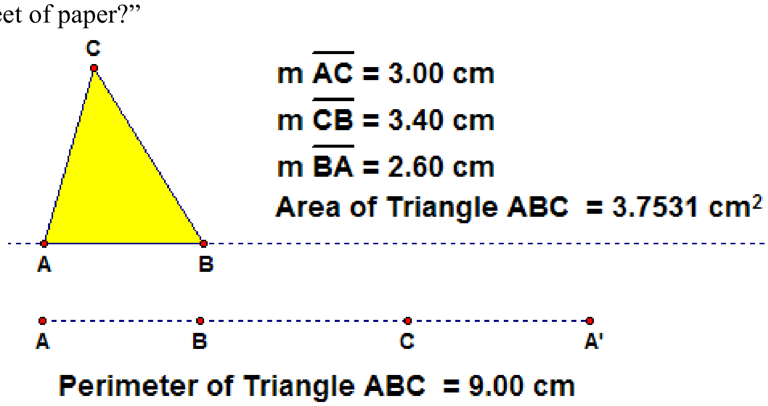


Figure 3: Finding a Triangle of Maximum Area

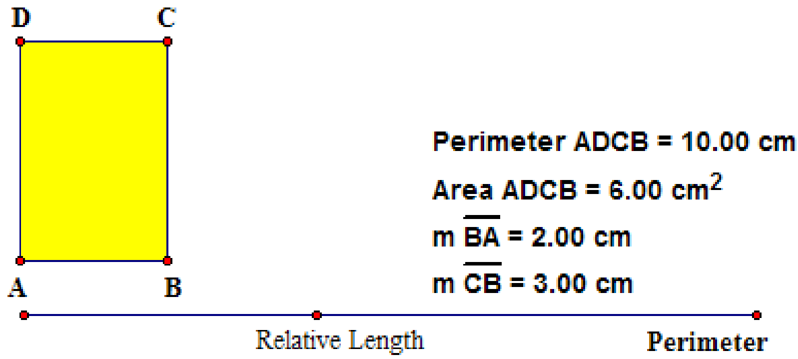


Figure 4: Finding a Rectangle of Maximum Area

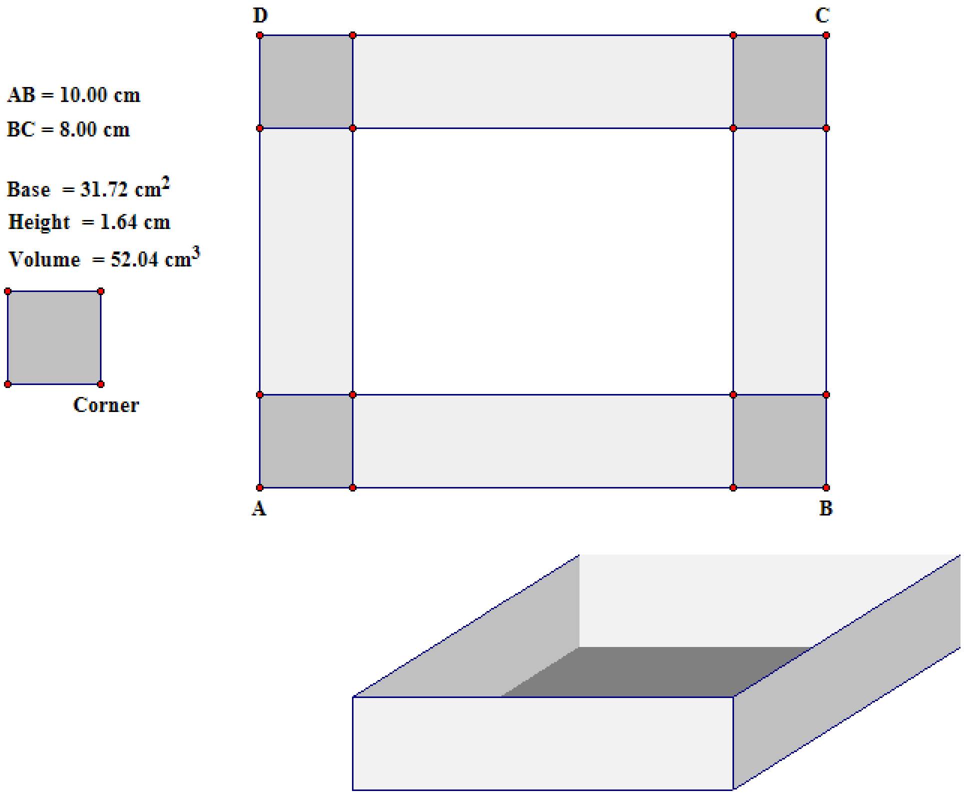


Figure 5: Finding an Open Top Box of Maximum Volume

Dynamic geometry may also be used to facilitate pattern discovery. For example, in Figure 6, a series of fraction buttons is used to color successively smaller regions of the square. This visualization of an infinite series is then associated with the mathematical notation for the series.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}\Rightarrow ?$$

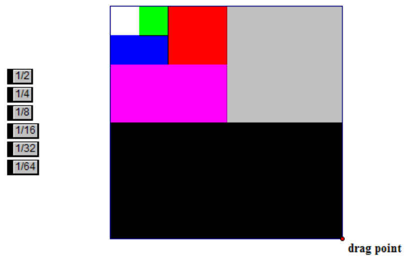


Figure 6: Visualizing an Infinite Series (1)

In Figure 7, a similar approach is used to visualize a different infinite series and to suggest its sum.

$$\frac{1}{4}+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\left(\frac{1}{4}\right)^4+\ldots\Rightarrow ?$$

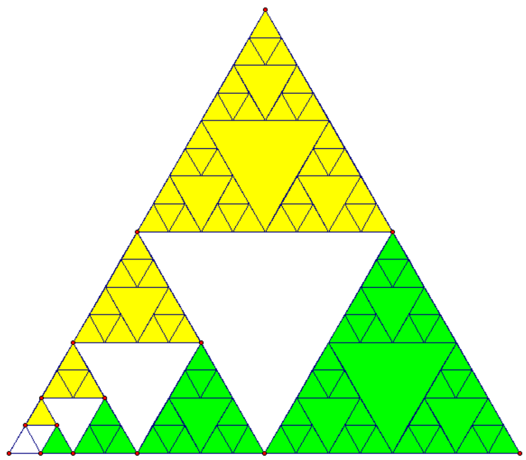


Figure 7: Visualizing an Infinite Series (2)

Observation ⇒ Conjecture ⇒ Testing ⇒ Belief

Dynamic geometry provides tools that may be used to facilitate construction and measurement of a wide variety of 2-dimensional objects. The challenge for teachers and students is to use these technologies as an extension of their imagination and to motivate and facilitate mathematical insight and discourse. Typically, students begin by making systematic observations of a construction provided by their instructor. Successful observations yield insights that are then stated as formal conjectures. These conjectures are tested using the same construction or model from which they were derived and either confirmed or rejected. Conjectures that are formulated, tested and confirmed in this manner become a part of students' mathematical belief system.

This sort of learning is not automatic for all students, however. Some students lack the prerequisite knowledge and skills necessary to fully participate in this learning sequence. Doing so involves more than mouse-dragging skills. Students must understand enough about the purpose and features of a given construction that questions about its transformation will have meaning. In particular, students must distinguish between the conditions (i.e., "If …") and the conclusions (i.e., "Then …") embodied by the construction. In a general sense, students must understand the difference between free objects (e.g., unconstrained points and lines often associated with condition information) and dependent objects. Specific to each construction, they must also understand which objects and free, which are not, and why this distinction matters during testing. Some students grasp these concepts and distinctions quickly. These students may be recognized by their proactive, systematic approach to observation and testing. Other students struggle knowing which objects to drag and how to interpret their observations. These students frequently attempt to mouse-drag dependent objects, which only adds to their frustration. When they do drag a free object, they often do so in a random, reactive manner rather than in a purposeful, systematic manner. Performance deficits of this sort require more than computer-skills to remedy. They are based on knowledge deficits. Can the interactive features of dynamic geometry be used to address these deficits? This is a fundamental challenge to the use of dynamic geometry as a tool of discovery and a topic worthy of thoughtful reflection on the part of teachers and researchers.

Whenever a discovery is made, it should be expressed in a formal conjecture that is both geometrically and grammatically precise. In doing so, students must call on their knowledge of geometric terminology, objects, concepts, notation and theorems. Students who lack a formal geometric vocabulary and who have difficulty understanding the conditional nature of geometric constructions and theorems are likely to struggle formulating a suitable conjecture. They may also have difficulty understanding conjectures posed by other students. Structuring classroom discourse to address this sort of difficulty poses yet another creative opportunity for teachers and researchers.

Once a conjecture is confirmed by testing, most students believe it to be true and immediately seek confirmation from the teacher. In our experience, this is an authentic and engaging context in which to focus student attention on reasoning and proof. One method for doing so makes use of the Properties option found in the Edit Menu of *The Geometers Sketchpad*.

Motivating and Facilitating Reasoning and Proof

When an object is selected in *The Geometers Sketchpad*, its dependencies relative to other objects in the construction may be seen by selecting Properties in the Edit Menu. For instance, in Figure 8, line k is constructed perpendicular to line j through point A.

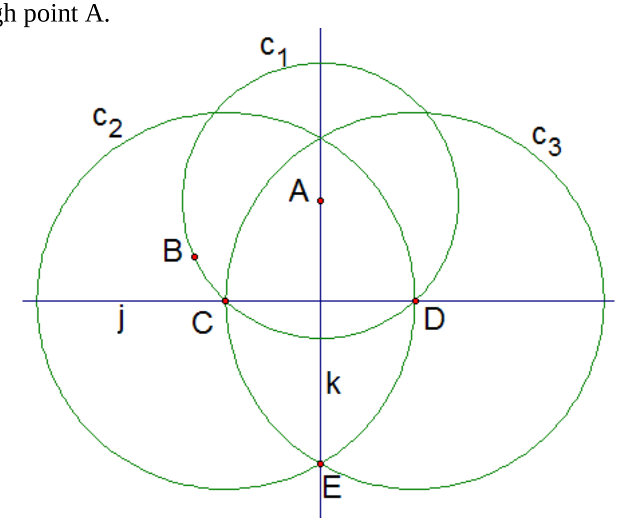


Figure 8: Construction of a perpendicular to a line through a given point

When line k is selected and it properties displayed (See Figure 9), the Properties dialog box shows that line k has two *parents*, point E (which is determined by an intersection) and point A (which is a given point) and no *children*. In this context, the term *parent* is used to indicate that line k is dependent on points E and A. Since no objects depend on line k, it has no *children*.

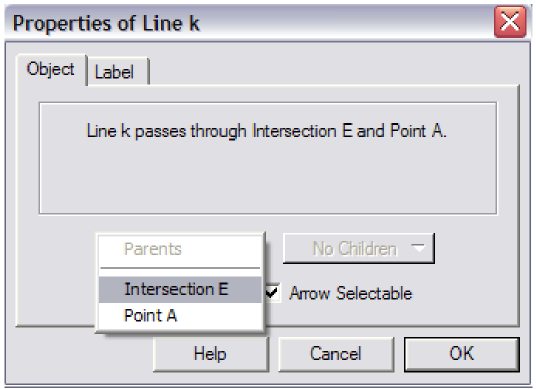
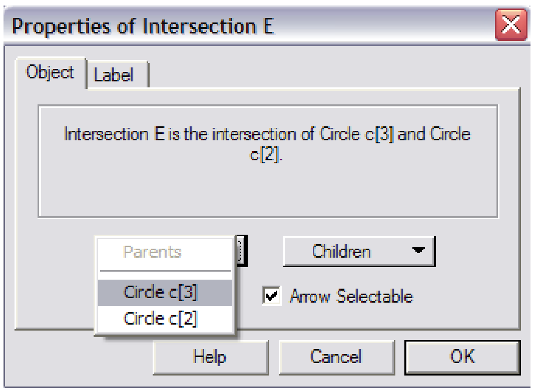
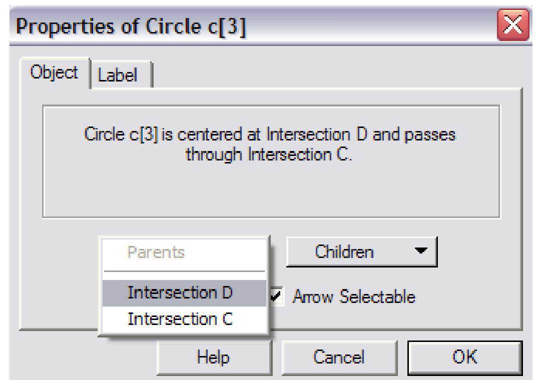


Figure 9: Step 1 [Backwards]

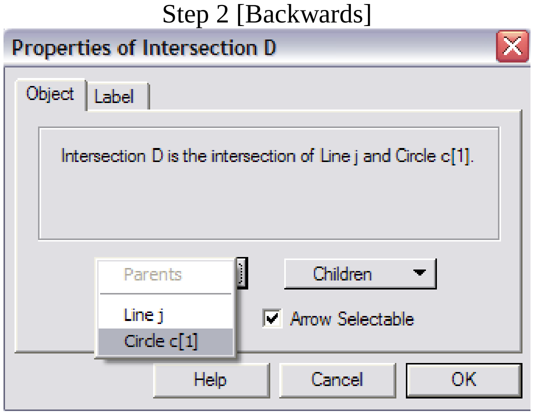
What are the parents of Intersection E? Of the other objects used in the construction? The sequence of Properties boxes seen in Table 1 reveals a sequence of dependencies that extends backwards to the initial conditions of the construction.



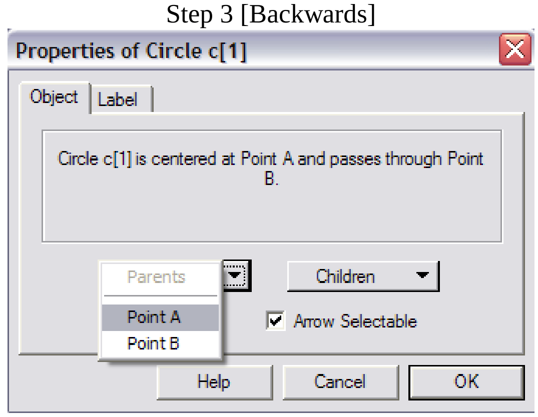
Step 2 [Backwards]



Step 3 [Backwards]



Step 4 [Backwards]



Step 4 [Backwards]

Table 1: Backward Dependencies

These backwards dependencies are summarized in Table 2. The table is meant to be read from the top down. For instance, line k is dependent on point A and E. Point A is free to move. Point E is dependent on circles c_2 and c_3. And so on. It is instructive to have students create tables of this sort from *Geometers Sketchpad* models created by the instructor. The point of this sort of exercise is to help students grasp the nature of logical dependency as they review the properties of the geometric objects used in the construction.

Parent	Child	Comments
k	A, E	Line k is determined by points A and E. Point A is free to move.
E	c_2 , c_3	Point E is determined by the intersection of circles c_2 and c_3.
c_2	C, D	C is the center; D determines the radius.
c_3	D, C	D is the center; C determines the radius.
C	j, c_1	Point C is determined by line j and circle c_1. Line j is free to move.
D	j, c_1	Point D is determined by line j and circle c_1. Line j is free to move.
c_1	A, B	Point A is free to move. Point B is free as long as circle c_1 intersects line j.

Table 2: Backwards Dependencies

Once students are able to "read" a construction in this manner, they may consider the "forward" version of the same dependencies (See Table 3). With careful consideration to the reasoning, this sequence of dependencies may be seen as the basis for a proof of the construction. For students who have no experience or success formulating their own proofs, practice of this sort may be used to reinforce attention to the properties of geometric objects, definitions, theorems, and reasoning itself.

Construct circle c_1	Center = A. Radius = AB. The circle must intersect line j.
Plot point D	Intersection of line j and circle c_1.
Plot point C	Intersection of line j and circle c_1.
Construct circle c_3	D is the center; C determines the radius.
Construct circle c_2	C is the center; D determines the radius.
Plot point E	Intersection of circles c_2 and c_3.
Construct line k	Containing points A and E.

Table 3: Forwards Dependencies

Discussion

For many people, the origins of mathematical knowledge are beyond comprehension. In the hands of a skillful teacher, dynamic geometry technologies may be used to bring students to the brink of mathematical discovery for the first time in their lives. The actual content of their first discovery isn't important in itself. What is important is that students realize that they are capable genuine mathematical discovery. This discovery changes lives. In our experience, this realization empowers students and raises their expectations about what they can and should expect of themselves as students of mathematics.

For many students, the learning sequence Observation ⇒ Conjecture ⇒ Testing ⇒ Belief generates a genuine interest in finding logical support for the beliefs they have expressed in their conjectures. It also establishes reasoning and proof as legitimate additions to the sequence: Observation ⇒ Conjecture ⇒ Testing ⇒ Belief ⇒ Informal Reasoning ⇒ Formal Proof. Using the Properties option in Edit menu of *The Geometers Sketchpad*, students may "peel back" the layers of logic in geometric models in an orderly manner, noting dependencies as they go. These dependencies may be seen from two perspectives:

- Parent-child relationships viewed "forwards" relative to the order of operations may be thought of as benchmarks in the construction, justification and/or validation of a model;
- Parent-child relationships viewed "backwards" relative to the order of operations may be thought of as dependencies and/or contingencies in the model.

Interestingly, there need be no "arrow of time" (i.e., preferred direction) to these observations, as they make sense seen from either perspective. This is true because many mathematical relationships are not fundamentally "cause and effect" in nature. This realization will come as a surprise to students who view mathematics primarily as a set of procedures. It is also a meaningful context in which to extend the discovery process to include reflection on the nature of mathematical reasoning and proof.

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