# The Mathematics of Perspective: An Introduction to the Cross Ratio

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## Introduction

Even though the world is a finite physical space, an infinite number of perspectives exists on the world's people, places, structures, and processes. Artists, architects, and designers specialize in creating and communicating perspectives that reflect their individual interests and values. In many cases, creating these perspectives involves representing three-dimensional objects on two-dimensional surfaces such as canvases, illustration boards, computer monitors, or movie screens. Since the Renaissance, artists, architects, scientists, and mathematicians have been developing concepts and procedures to guide this process. This article focuses on a mathematical concept underlying many of these efforts, the cross ratio.

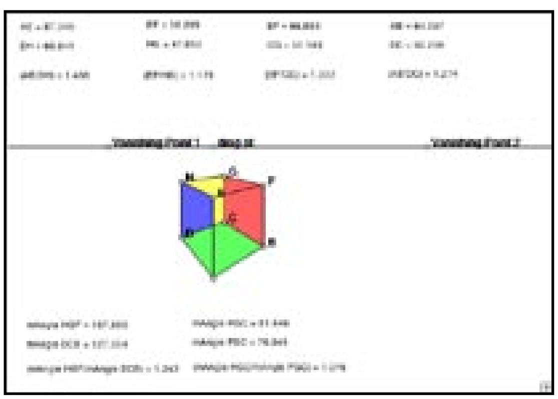

**Figure 1** Perspective View of a Cube

## History

Projective geometry was developed by Renaissance artists and architects faced with a common problem: The geometric features of buildings and landscapes appear to change depending on one's point of view, or perspective. For example, lines that are known to be parallel appear to converge in the distance; the near side of a rectangular wall appears to be longer than the far side; and, seen from the bottom, steps at the bottom of a staircase appear larger than steps at the top. Effects of this sort are an inevitable consequence of projecting a three-dimensional object onto the human retina, a two-dimensional surface. A number of effects of this sort are apparent in **figure 1,** a perspective view of a cube, created using the Geometers Sketchpad and converted into a java applet using the Java Sketchpad. Using the mouse pointer, the drag point and two vanishing points shown in the figure may be repositioned on the horizon line. Moving the drag point repositions the observer relative to the cube's closest vertical faces. Repositioning the vanishing points makes the cube appear nearer or farther from the observer in the directions of the vanishing points. Moving point E changes the apparent height of the cube.

See Student Activity: Perspective View of a Cube Page 12

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**Leone Battista Alberti** was the first person to develop a systematic approach to perspective drawing. Alberti was born in Genoa, Italy, in 1404, the second son of a wealthy merchant. Alberti acquired his early education at Gasparino Barzizza's Gymnasium in Padua, graduating in 1421. From there, he went to the University of Bologna, where he studied law but excelled in literature and geometry. After graduating in 1428, Alberti worked for several years as a secretary in the Papal Chancery in Rome, writing biographies of the saints in elegant Latin and traveling extensively in Europe. By 1432, he was living in Florence, working with famed artists Brunelleschi and Donatallo. He also collaborated with Toscanelli on the development of maps that Columbus later used on his first voyage in 1492.

Of his many interests, Alberti was most enthusiastic about the development of a geometrical basis for perspective drawing. In 1434 he wrote Della Pittura, the first written exposition on how to add a realistic third dimension to paintings. In it he said, "Nothing pleases me so much as mathematical investigations and demonstrations, especially when I can turn them to some useful practice drawing from mathematics the principles of painting perspective and some amazing propositions on the moving of weights." Alberti's influence on the development of Renaissance painting was significant and long-lasting. Leonardo da Vinci is known to have taken passages directly from Della Pittura and incorporated them into his Trattato (a common and legal practice at the time). He also greatly extended Alberti's ideas and techniques. Alberti's ideas did more than advance the development of painting, however; they revitalized the study of geometry. With the introduction of a new geometry, students of science were suddenly able to view Euclidean geometry as "a geometry" rather than "the geometry." Few thinkers have enabled such a profound change in perspective. Alberti died in Rome in 1472.

**Figure 2** depicts Alberti's method for creating a perspective view. In this figure, the array of tiles seen in the Map View is converted into the array seen in the Perspective View. In the Perspective View, the eye is naturally drawn into the center of the image by the apparent convergence of parallel lines in the distance and by the realistic scaling of the tiles that are positioned at different distances from the observer. Because only one vanishing point exists, the view is called one-point perspective.

Alberti knew that his method worked, in the sense that it produced a realistic view of three-dimensional objects. He did not know, however, why it worked. This article presents the key mathematical concept behind Alberti's method, the cross ratio.

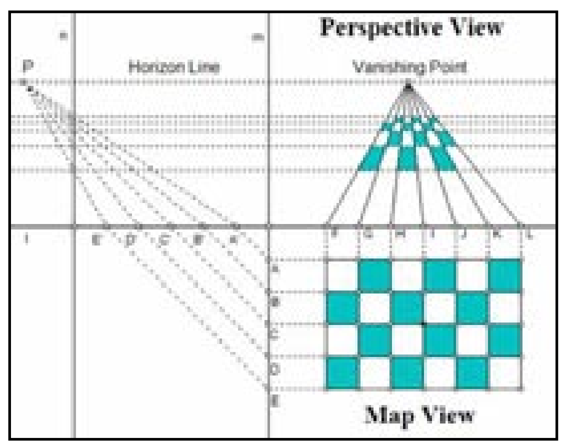

**Figure 2** Alberti's Method

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## Dividing Space: Collinear Points

Artists and architects of the Renaissance succeeded in developing drafting techniques that correctly and realistically represented three-dimensional objects on two-dimensional surfaces. These types of drawings systematically resize line segments and angles to create the illusion of depth. For example, in **figure 3**, the edges and angles of the cube, which we understand are equal in three-dimensional space, are resized to create the illusion of depth. In searching for an underlying mathematical principle for perspective drawing, one might hope to find a simple ratio expressing the relationship between the lengths of segments HG and EF, GC and FB (as well as other measurements) that is consistent from one perspective to another. Using the mouse pointer, it is possible to reposition certain points in figure 3, changing the apparent features of the cube. For instance, if you move the drag point along the horizon line, all of the segment measurements and their indicated ratios change. Consequently, none of those features are invariant under this transformation. On the other hand, when the vanishing points are moved, the segment lengths change but their ratios and respective angles do not change. Because this transformation preserves segment ratios and angles, all of these perspective views are geometrically similar. Finally, moving point E changes all of the measurements and their ratios.

See Student Activity: Creating the Illusion of Depth Page 12

The idea that some ratio of segments or angles should be invariant under perspective transformations is appealing, but, no such ratio is apparent as a result of this investigation. Surprisingly, however, this idea is only one step short of the truth. The sought-for relationship is not a simple ratio but a ratio of ratios called the cross ratio. Ignoring conventions concerning the sign of directed line segments, the following definition suffices for the purpose of this paper.

**Definition** For any four collinear points A, B, C, and D (See fig. 4), the cross ratio characterizes the partition of segment AD by points B and C, where

$$R = \frac{BA/BC}{DA/DC}$$

See Student Activity: Cross Ratio Computed in Terms of Segments Page 13

In projective geometry, a set of collinear points such as A, B, C, and D is called a pencil of points. In defining the cross ratio, it is useful to consider this pencil of points in a pair-wise manner. In **figure 4**, points A and C may be thought of as reference points, with points B and D as variable points. The distance from point B to each of the reference points may be expressed as the ratio BA/BC. Similarly, the distance from point D to each of the reference points may be expressed as the ratio DA/DC. The cross ratio is defined to be the ratio of these ratios and is characteristic of the manner in which points B and C partition (i.e., divide) segment AD. For instance, in Figure 4, the cross ratio computed in terms of segments BA, BC, DA, and DC is given as 0.331. Moving point A, B, C, or D changes the length of two segments and their ratio, leaving the other measurements and ratio the same. As a result the cross ratio changes, reflecting a change in the partition of segment AD.

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**Figure 4** also shows line segments drawn from point P to points A, B, C, and D. In projective geometry, a concurrent set of segments such as PA, PB, PC, and PD is called a pencil of segments. When a line segment is drawn across this pencil of segments, we obtain the projection of points A, B, C, and D on that segment in the form of points E, F, G, and H. Likewise, points I, J, K, and L may be thought of as a different projection of points A, B, C, and D. **Figure 4** shows the cross ratios for the partitions of segments EH and IL. Amazingly, both of these cross ratios are equal to the cross ratio computed in terms of points A, B, C, and D. Furthermore, repositioning points P and Q does not affect this result.

In every case, the cross ratio is invariant. It is only when the positions of point A, B, C, and D change relative to one another that their cross ratio changes. So, the cross ratio of a collinear set of 4 points may be computed based on the distances between the points themselves or on the basis of their projections onto another line.

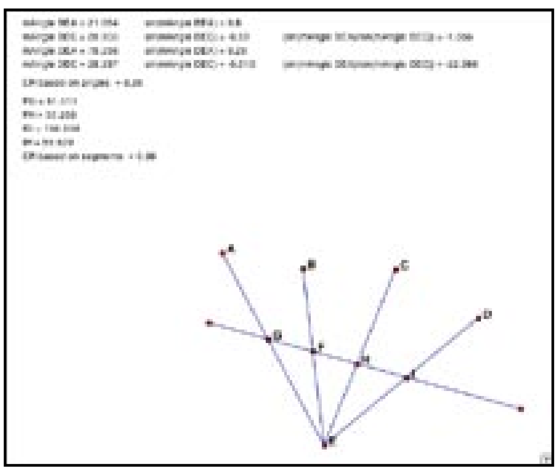


**Figure 3** Creating the Illusion of Depth

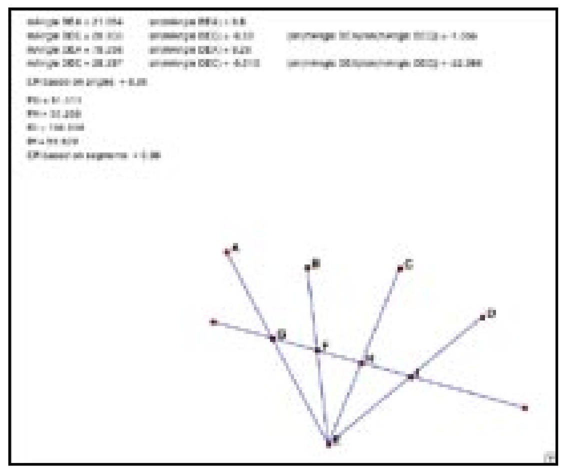


**Figure 4** Cross Ratio Computed in Terms of Segments

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## Dividing Space: Non-Collinear Points

We now consider the manner in which a set of rays divides space. Consider rays EA, EB, EC, and ED in **figure 5**. Note that points A, B, C, and D are not collinear. In this case, a cross ratio is computed based on the angles formed.

Definition For any set of lines EA, EB, EC, and ED, the cross ratio R characterizes the partition of $\angle DEA$ by lines EB and EC, where

$$R = \frac{\sin\angle BEA / \sin\angle BEC}{\sin\angle DEA / \sin\angle DEC}$$

See Student Activity: Cross Ratio for Non-Collinear Points Page 14

**Figure 5** reports the cross ratio using both angles and segments. The fact that both methods produce the same result is easily verified by experimenting with the applet, or better yet, the GSP file from which the applet was created. The value in this finding is seen in the versatility of the approach: One may compute the cross ratio using the best or most convenient data available. Note, however, that the cross ratio is not independent of the point of view of the observer (point E) unless points A, B, C, and D are collinear.

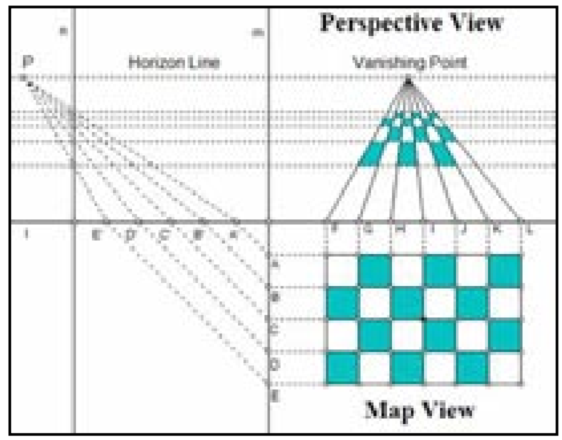


**Figure 5** Cross Ratio for Non-Collinear Points

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## Projective Transformations

In the examples considered so far, the planar figures are shown in what is sometimes called a map view, as if you were looking down on the figure from directly overhead. This is the conventional manner frame of reference for figures in plane geometry, so it is from this perspective that we normally measure segments and angles.

We now turn our attention to perspective views of planar figures and ask the question, "If we measure the segments and angles in a perspective view of a figure and use those measurements to compute a cross ratio, will the result be the same as or different from that obtained using measurements based on a map view of the same figure?" This situation is illustrated in **figure 6**, a construction based on Alberti's method. Clearly, the red lines seen in the map view (lower) appear to be different in length than and form different angles than the red lines seen in the perspective view (upper). Using the Geometers Sketchpad, we may measure the segments and angles seen in each view and use those measurements to compute cross ratios. Using the mouse pointer, we may then change the perspective view in order to compare cross ratios associated with different projections. To do so, move the vanishing point and/or point P along the horizon line. In all cases, the cross ratios remain unchanged. Another way of expressing this idea is, "The cross ratio is invariant under projective transformations." This fact has far-reaching consequences in computer graphics, architecture and design, and scientific visualization.

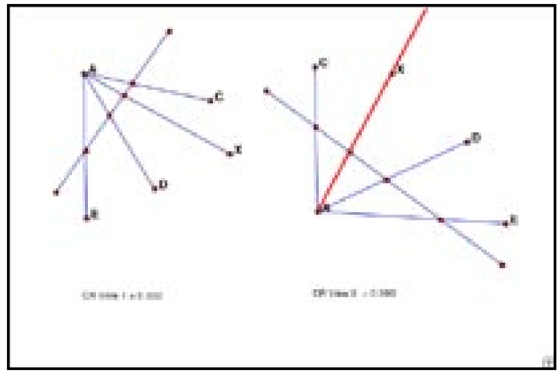


**Figure 6** Perspective and Map Views of the Same Figure

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## Martian Canyon System: Two Views

Figure 7 shows a perspective view of a canyon system on Mars (image courtesy of NASA). The observer is positioned on the south side of the canyon, facing north. Selected landmarks are labeled with the letters A, B, C, X, D, and E. Superimposed on this image is a set of lines used to compute the cross ratio determined by rays AC, AX, AD, and AE. For reasons that will become apparent later, this cross ratio is computed indirectly using the line segments associated with points P1, P2, P3, and P4.

Figure 8 is an abstraction of the scene in figure 7. Using the mouse pointer, move the Drag Point so that line AP3 intersects point X. Doing so results in a cross ratio of 0.38, the same value in figure 7. Notice that the cross ratio associated with any other positioning of line AP3 is different.

See Student Activity: Perspective and Map Views of the Same Figure Page 14

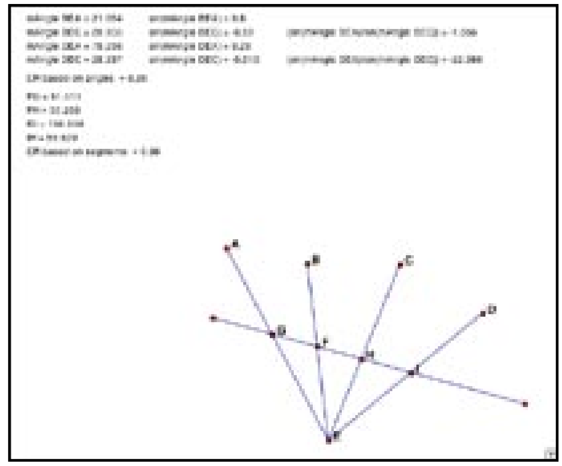

**Figure 8** Finding a Cross Ratio Associated with View 1

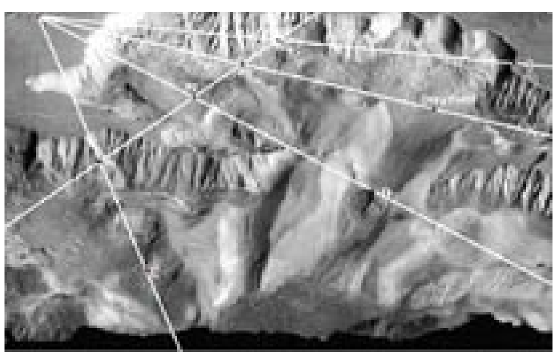

**Figure 7** View 1 of a Canyon System on Mars

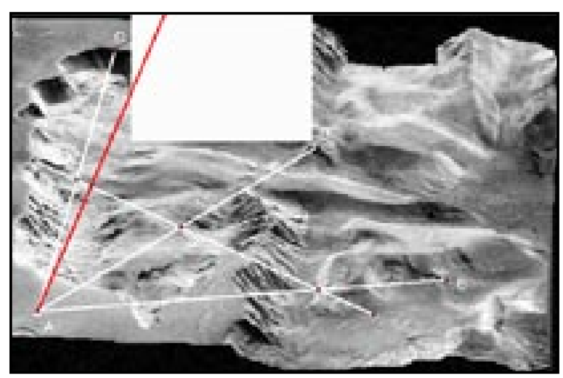

**Figure 9** View 2 of the Same Canyon System

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Because the cross ratio is invariant under projective transformations, viewing any pencil of points or rays from a different perspective should yield the same cross ratio. In particular, if you use a different perspective view of the same canyon system and recalculate the cross ratio based on measurements of the same rays or segments as seen from that perspective, you should obtain the same cross ratio. Figure 9 shows such a view with a white box superimposed on the region containing point X. Where should an observer be located in order to achieve this perspective? Using your knowledge of the cross ratio, where would you position the red line so that it is certain to contain point X?

Figure 10 is an abstraction of the scene in figure 9. Using the mouse pointer, reposition the red line in figure 10 so that the cross ratio equals 0.38. Why does this cross ratio guarantee proper placement of the red line?

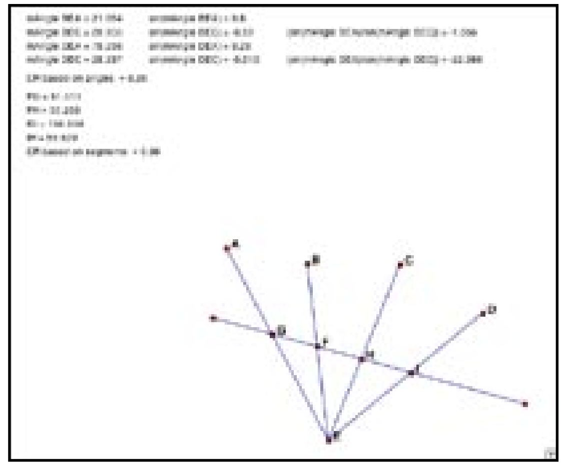

**Figure 10** Using the Cross Ratio to Position Line AX in View 2

Figure 11 abstracts and generalizes the specific problem that figures 7 - 10 illustrate. Assume that the two views correspond to two perspectives of a geographic region on Mars. In both views points A, C, X, D, and E represent the same landmarks. In View 1 (on the left), position point X so that the indicated cross ratio is approximately 0.90. Then adjust the red line in View 2 (on the right) to the same value. What does the red line in View 2 tell you about the location of point X? How could you use the ideas demonstrated here to identify the actual location of point X in View 2? What additional information would you need?

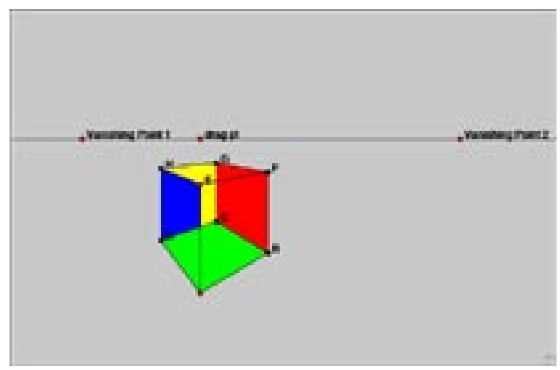


**Figure 11** General Problem

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## Martian Canyon System: In the Cross Hairs

In order to locate point X in View 2, we must know its direction from two reference points. In figure 12, both point A and point B serve as reference points. Using the procedures presented in "Martian Canyon System: Two Views", we may find a cross ratio for both the black and white line sets.

In figure 13, each red line is positioned so that **lines** AX and BX, respectively, result in the same cross ratios obtained in View 1. In effect, you locate point X with a "cross hair."

One aspect of this analysis introduces a significant source of error: Although all the lines are assumed to lie in the same plane, the landscape features marked A, B, C, X, D, and E do not. What effect might this assumption have on the accuracy of the result?

Cartographers (map makers) have used a low-tech version of this procedure called the paper-strip technique for decades. Cartographers gather data from a variety of sources, including surveys, photographs, and satellite images. The cartographer's job is to create unbiased map views that accurately identify significant landmarks (both natural and man-made), longitude and latitude, and other information (such as, elevation, land use, population statistics, and so on). The paper-strip technique removes the bias inherent in the perspective views of photographic and satellite data sources. The steps implementing this technique as applied to the Martian canyon system are as follows:

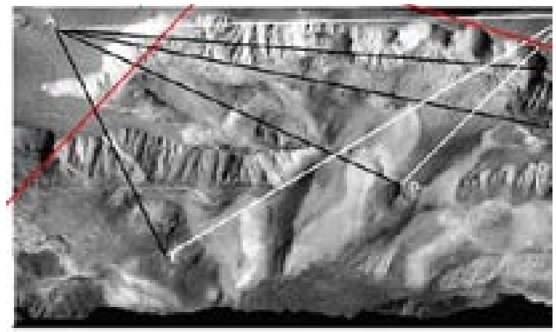

**Figure 12** Two Reference Points

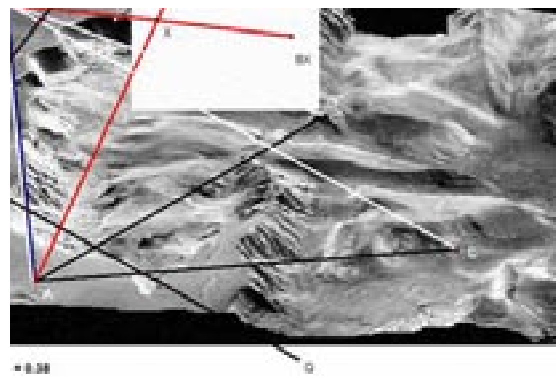

**Figure 13** Locating Point X

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1: Draw a set of lines from point A to points C, X, D, and E in View 1 (See fig. 14). Then lay a strip of paper across these lines and mark their intersection points.

2: Draw lines AC, AD, and AE in View 2 (See fig. 15). Then position the paper strip from Step 1 so that the marks line up as shown. The unused point determines the position of line AX.

3: Repeat Step 1, drawing the set of lines from point B and using a second paper strip.

4: Repeat Step 2, using the second paper strip to determine the position of line BX.

5: Point X is located at the intersection of lines AX and BX.

Why does the paper-strip technique work? In a mathematical “shell game” of sorts, a cross ratio based on rays from point A is converted to a cross ratio based on segments marked on the paper strip. We then use the strip to create a pencil of rays with the proper cross ratio as seen in the second view of the canyon. This procedure yields the direction to point X from point A. When the procedure is repeated using point B as the reference point, a direction from point B to X is determined. The mysterious point X is then located at the intersection of these two directional rays.

Print out the images in figures 16 and 17 on Page 16, then use them to practice the paper-strip technique.

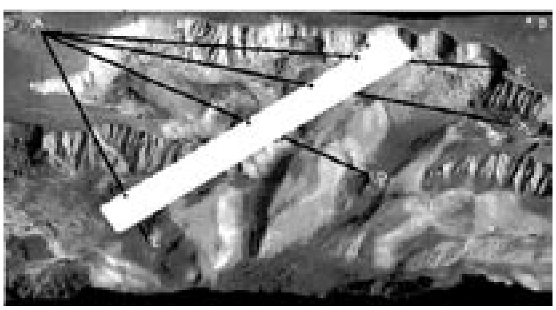

**Figure 14** Step 1

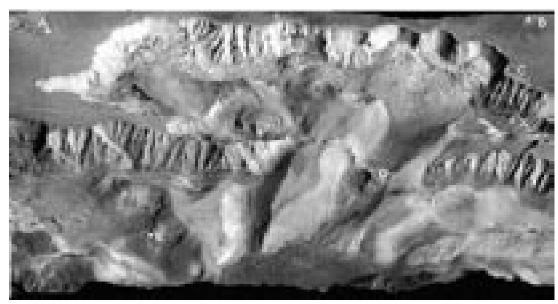

**Figure 16** View 1 Practice Image

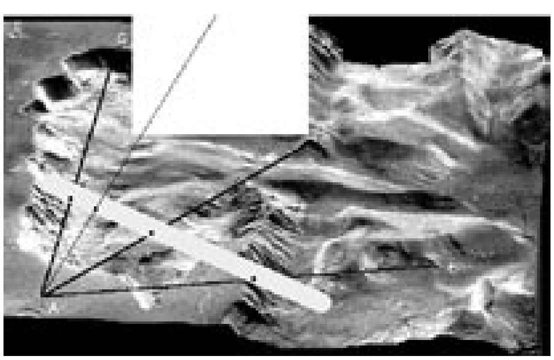

**Figure 15** Step 2

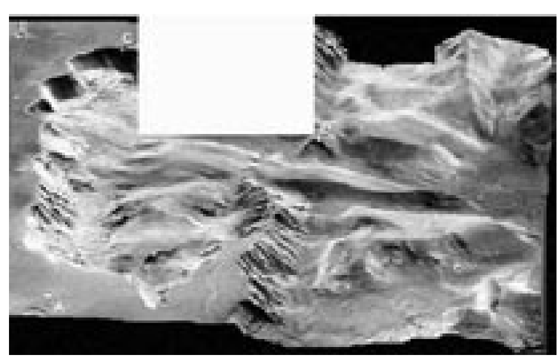

**Figure 17** View 2 Practice Image

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## Summary

Every time you look at a building from a different location, you see it from a different perspective. In addition to revealing different details of the building, each perspective provides important visual clues about the relative distance to objects in our field of view. These clues arise in the form of apparent distortions. For example, features that are known to be of equal length, area, or angle appear to be different when seen in perspective. Far from misleading, these distortions are essential to our depth perception and understanding of our surroundings.

This article focuses on an aspect of perspective that we cannot experience directly, a division of space that is independent of our perspective or point of view. Mathematicians call this division the cross ratio. As this article discussed, the cross ratio—

- is a ratio of ratios;
- may be computed using collinear segments;
- may be computed using concurrent angles;
- is independent of the posiion of the observer;
- is invariant under projective transformations;
- is the basis for the paper-strip techique that cartographers used; and
- is an important mathematical basis for 3-D computer graphics.

The MacTutor History of Mathematics Archive. http://turnbull.dcs.st-and.ac.uk/history/

# references

The Geometer's Sketchpad, Emeryville, Calif. Key Curriculum Press, 2002.

Thomas, D. (2002). Modern Geometry. Pacific Grove, Calif. Brooks Cole Publishing Company, 2002.

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# student activities

## Student Activity: Perspective View of a Cube

1. Using the mouse key, move the drag point in figure 1 first to the right, then to the left. How does the view of the cube change?

   ______________________________________

   ______________________________________

2. Using the mouse key, move Vanishing Point 1 and Vanishing Point 2. Describe the apparent changes in your view of the cube.

   ______________________________________

   ______________________________________

3. Using the mouse key, move Point E. Describe the apparent changes in your view of the cube.

   ______________________________________

   ______________________________________

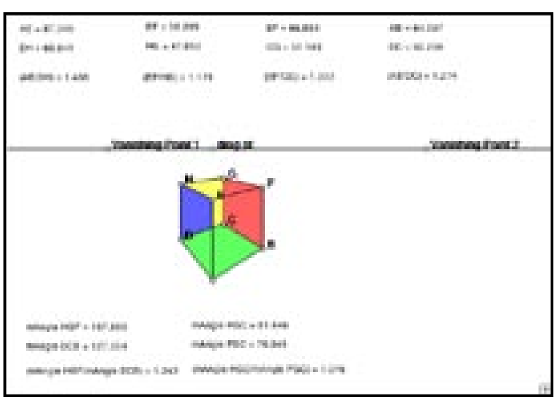

**Figure 1** Perspective View of a Cube

## Student Activity: Creating the Illusion of Depth

1. Move the drag point along the horizon line and note how the segment lengths and their ratios change. Does any segment or ratio of segments remain constant for a fixed set of Vanishing Points?

   ______________________________________

2. Move the drag point along the horizon line and note how the angles and their ratios change. Does any angle or ratio of angles remain constant for a fixed set of Vanishing Points?

   ______________________________________

3. Move the Vanishing Points along the horizon line. Which measurements or ratios remain fixed?

   ______________________________________

   ______________________________________

4. Move point E vertically. Which measurements or ratios remain fixed?

   ______________________________________

   ______________________________________

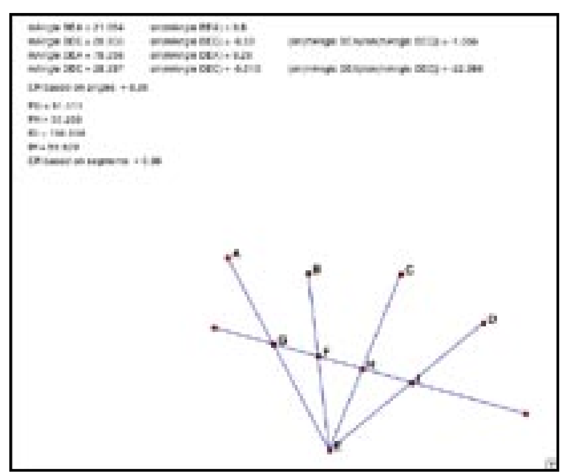

**Figure 3** Creating the Illusion of Depth

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# student activities

## Student Activity: Cross Ratio Computed in Terms of Segments

1. Using the pointer, reposition points A, B, C, and D. What happens to the cross ratio computed in terms of segments BA, BC, DA, and DC?

    ____________________________________

    ____________________________________

2. How does the cross ratio computed in terms of segments BA, BC, DA, and DC compare to the cross ratio computed in terms of the segments joining points E, F, G, and H? In terms of the segments joining points I, J, K, and L?

    ____________________________________

    ____________________________________

3. Leaving points P, Q, A, B, C, and D fixed, reposition the segment containing points E, F, G, and H. What happens to the cross ratios computed in terms of those segments?

    ____________________________________

    ____________________________________

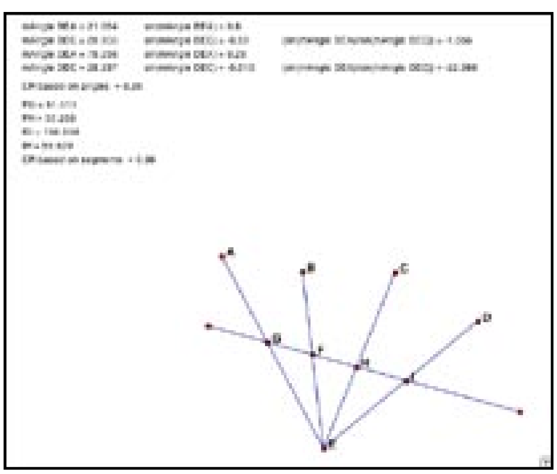

**Figure 4** Cross Ratio Computed in Terms of Segments

4. Leaving points P, Q, A, B, C, and D fixed, reposition the segment containing points I, J, K, and L. What happens to the cross ratios computed in terms of those segments?

    ____________________________________

    ____________________________________

5. Leaving points A, B, C, and D fixed, reposition points P and Q. Do the cross ratios computed in this manner depend on the positions of points P and Q?

    ____________________________________

    ____________________________________

6. State a conjecture about the cross ratio based on your observations.

    ____________________________________

    ____________________________________

    ____________________________________

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# student activities

## Student Activity: Cross Ratio for Non-Collinear Points

1. Reposition point E, moving line XY as necessary. Compare the cross ratio computed based on the angles formed with the cross ratio computed using the segments created by points G, F, H, and I. State a conjecture comparing the cross ratio determined using angles and the cross ratio determined using segments.

   ____________________________________

   ____________________________________

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2. Reposition points A, B, C, and D so that they coincide with points G, F, H, and I, respectively. Then move Point E. What do you notice?

   ____________________________________

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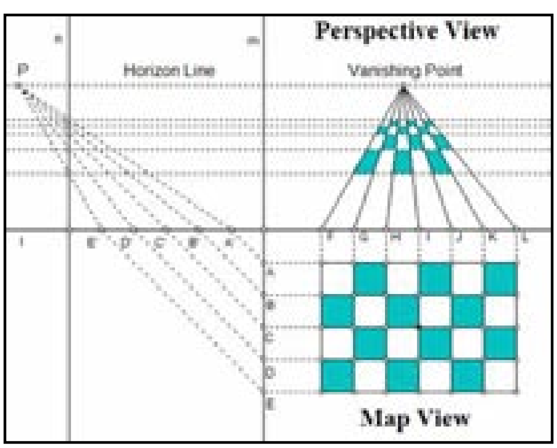


**Figure 5** Cross Ratio for Non-Collinear Points

## Student Activity: Perspective and Map Views of the Same Figure

1. Using the pointer, drag around the Vanishing Point and point P along the horizon line. How does the perspective view of the tile floor change?

   ____________________________________

2. How does your position as an observer appear to change from one projection to the next?

   ____________________________________

   Using the segment-method, the cross ratio is computed based on the manner in which the tiles divide the heavy red line in both the Map View and the Perspective View.

   ____________________________________

3. What do you notice about these two cross ratios?

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4. State a conjecture about the cross ratio as computed from perspectives other than map views.

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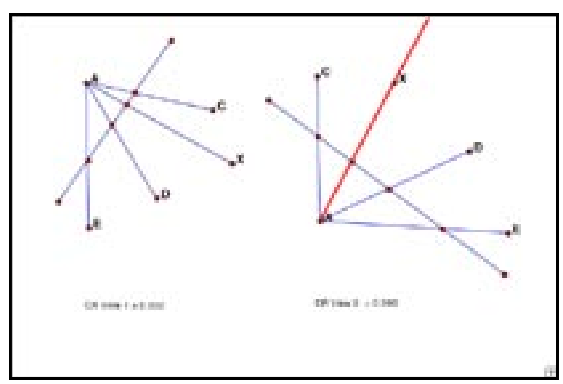

**Figure 6** Perspective and Map Views of the Same Figure

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# student activities

### Student Activity: Perspective and Map Views of the Same Figure

Use the mouse key to move the Drag Point so that line AP3 intersects point X.

1. If the top of the scene is north, where should the observer be located in order to achieve this perspective?

   ____________________________________

2. What cross ratio is obtained?

   ____________________________________

3. If you relocate line XY, what happens to the cross ratio?

   ____________________________________

   ____________________________________

4. If you had a map view of the same canyon system and recalculated the cross ratio based on measurements of that image, what value would you obtain?

   ____________________________________

5. If you had a different perspective view of the same canyon system and recalculated the cross ratio based on measurements of that image, what value would you obtain?

   ____________________________________

   ____________________________________

Figure 9 shows a different view of the canyon system, with a white box superimposed on the region containing point X. Your objective is to deduce the location of point X using your knowledge of the cross ratio. Using the mouse key, reposition the red line in Figure 4 so that AX points in the correct direction.

____________________________________

____________________________________

6. Where should the observer be located in order to achieve this perspective?

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7. How do you know when the red line is correctly positioned?

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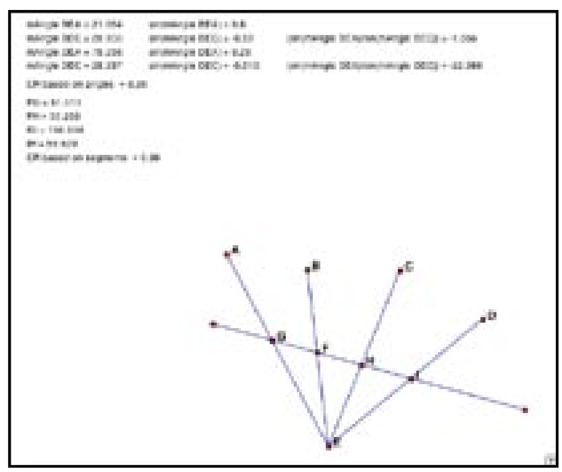

**Figure 8** Finding a Cross Ratio Associated with View 1

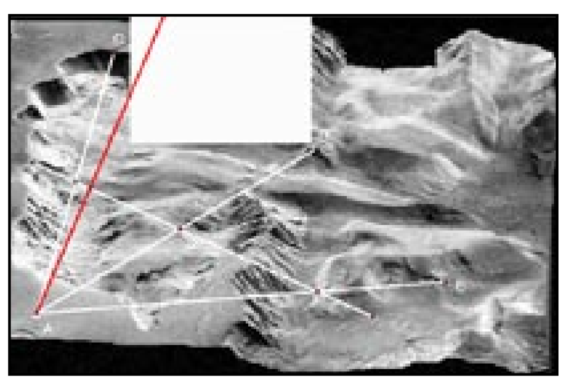

**Figure 9** View 2 of the Same Canyon System

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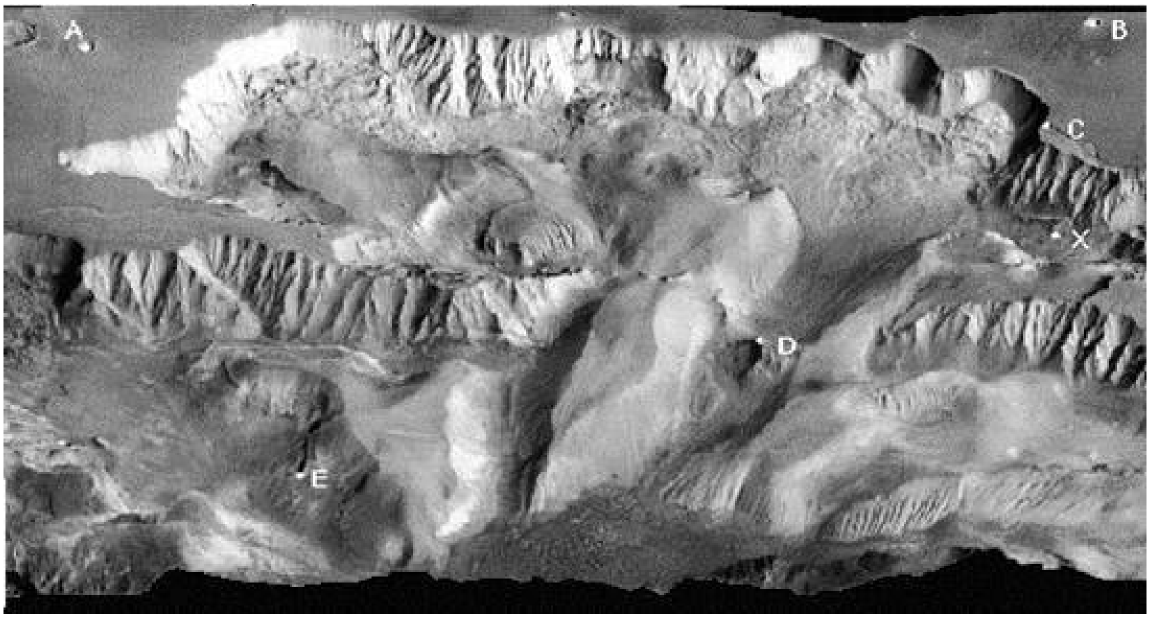


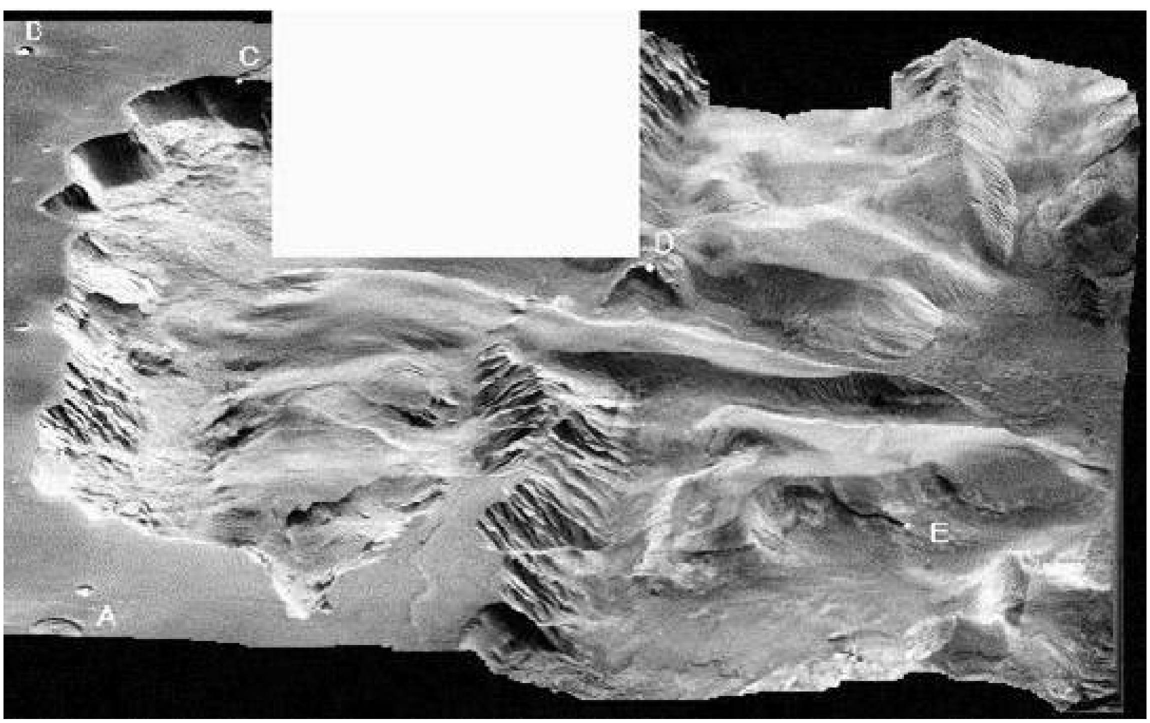


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